Technical Note: Precision Focusing and Resource Allocation via the Fast Walsh–Hadamard Transform

1. Background and Motivation

Modern systems frequently operate under **tight precision constraints**:

- Low-bit weights in neural networks (e.g. 8–16 bits)
- Memory bandwidth limits
- Quantized inference pipelines

At the same time, signals and parameters often require **non-uniform precision**:

- Some directions/features are more important than others
- Uniform quantization wastes capacity on irrelevant components

A natural question arises:

> Can we **reallocate precision** across dimensions in a controlled way?

This note describes how linear transforms—particularly the Fast Walsh–Hadamard Transform—provide a simple and efficient mechanism for doing exactly that.

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2. Transform as a Change of Basis

$$H_1 = [1]; \quad H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix};$$

$$H_4 = H_2 \otimes H_2 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$H_8 = H_2 \otimes H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Given $x \in \mathbb{R}^n$, the FWHT computes:

$$y_k = \sum_{i=1}^{n} s_{k,i}\, x_i \quad \text{with } s_{k,i} \in \{+1, -1\}$$

Key properties:

- Linear and invertible
- O(nlogn) complexity
- Dense, global mixing (each output depends on all inputs)

Interpretation:

> The transform maps from **coordinate space** to a **pattern space** (Walsh basis).

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3. Core Mechanism: Coherent vs Incoherent Addition

Each output is a signed sum. Two regimes emerge:

- **Coherent addition**: terms align → magnitude O(n)
- **Incoherent addition**: cancellation → magnitude O(√n)

This distinction is the foundation of the “focusing” viewpoint.

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4. Focusing: Concentrating Precision into One Output

If the input is aligned with a single Walsh pattern:

- One output receives fully **constructive interference**
- All others cancel

Result:

- One coefficient ≈ n · a
- Others ≈ 0

Interpretation

> All degrees of freedom are **focused into a single channel**

This is directly analogous to:

- Matched filtering
- Beamforming
- Coherent gain in signal processing

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5. Precision Gain via Dynamic Range Expansion

With bounded inputs (e.g. 8-bit integers):

- Summation increases dynamic range by ~$\log_2 n$ bits

Example:

- (n = 256), 8-bit inputs
- Focused output ≈ 16-bit dynamic range

So:

> Many low-precision inputs can produce a **single high-precision output**

This is a **redistribution of precision**, not an increase in total information.

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6. Splitting Focus Across Multiple Outputs

If we focus on k outputs:

- Each receives partial coherent contribution
- Magnitude per output $\approx n/k$

Focused Outputs	Precision per Output
1	Maximum (~n)
2	~n/2
k	~n/k

Interpretation

> Precision becomes a **divisible resource** that can be allocated across outputs.

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7. Resource Allocation View

The transform induces a constraint:

- Total “coherent capacity” ≈ (n)

You can choose:

- Few outputs with high precision
- Many outputs with lower precision

This reframes the transform as:

> A **precision allocation mechanism over basis directions**

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8. Geometric Interpretation

With normalization, the FWHT preserves energy:

- Data lies near a **sphere** in high dimensions
- The transform rotates coordinates to the Walsh basis

Then:

- Focusing = movement along specific directions with large amplitude
- Splitting focus = distributing energy across multiple directions

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9. Quantization: Selecting What Survives

In practical systems:

- Outputs are quantized (e.g. 16-bit float, 8-bit integer)

Effects:

- Large coefficients → robust to quantization
- Small coefficients → lost or noisy

Thus:

> Focusing determines which information is preserved under finite precision

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10. Weak Learners Perspective

Let each input x_i be a **weak learner**:

- Low precision
- Noisy / low signal

The FWHT aggregates them:

Single-output focus

- All learners align
- One strong, high-SNR signal emerges

Multi-output focus

- Learners split across channels
- Multiple moderate-strength signals

Interpretation:

> The transform converts many weak signals into fewer strong, structured signals.

This connects naturally to:

- Ensemble methods
- Boosting
- Signal aggregation

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11. Projection and Compression

Selecting a subset of outputs (e.g. top-k):

- Projects onto a **low-dimensional subspace**
- Defined by Walsh patterns

This enables:

- Compression
- Feature extraction
- Approximate reconstruction

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12. Applications

(a) Variable Precision Neural Network Weights

- Apply FWHT to weight blocks
- Allocate precision unevenly across coefficients
- Store compactly
- Reconstruct approximately during inference

Benefits:

- Reduced quantization error for important components
- Improved robustness under low-bit regimes

(b) Activation Compression

- Transform activations
- Retain high-energy coefficients
- Quantize or discard the rest

(c) Approximate Computing

- Dynamically allocate precision based on importance
- Trade accuracy for efficiency in controlled ways

(d) Distributed / Associative Representations

- Information is spread across coefficients
- Robust to partial corruption
- Supports “holographic” storage

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13. Key Insight

> The Walsh–Hadamard transform allows **precision to be focused or distributed across outputs**, enabling controlled trade-offs between dimensionality and numerical accuracy.

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14. Limitations

- No increase in total information capacity
- Gains come from **error redistribution**, not elimination
- Effectiveness depends on signal structure (alignment with basis)

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15. Directions for Further Study

To explore these ideas more deeply, relevant areas include:

- **Transform coding** (e.g. DCT, Walsh transforms)
- **Rate–distortion theory** (from Information Theory)
- **Vector quantization and spherical codes**
- **Compressed sensing and random projections**
- **Hyperdimensional computing / holographic representations**
- **Mixed-precision and learned quantization in neural networks**
- **Hadamard codes and error-correcting codes**

You may also look into recent work on:

- Transform-domain neural network compression
- Block-wise orthogonal transforms for quantization
- Hardware-efficient Hadamard-based inference

(Some systems and papers use Hadamard-like transforms to improve quantization efficiency; terminology varies across implementations.)

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16. Summary (Intuition)

> The FWHT acts like a “precision lens”: it can concentrate many low-precision inputs into a few high-precision outputs, or distribute precision broadly across many

outputs. This enables flexible, structured control over how information survives under finite precision constraints.